UNIVERSIDAD SAN FRANCISCO DE QUITO USFQ

Colegio de Ciencias e Ingenierías

Higher-order corrections for the deflection of light around a massive object, and Friedmann equations for a time-varying G, Λ and c.

Proyecto de investigación

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RESUMEN

Obtenemos los términos de order superior para la desviación de la luz alrededor de un agujero negro utlizando el método perturbativo de Lindstedt-Poincaré para resolver la correspondiente ecuación de movimiento. Además, obtenemos aproximantes de Padé diagonales a partir de esta expansión perturbativa, y mostramos que estos se ajustan más a los datos numéricos. Adicionalmente, mostramos cómo estas aproximaciones se pueden utilizar para algoritmos de *raytracing*, en el modelamiento de la desviación de la luz alrededor de agujeros negros. También consideramos las ecuaciones de Friedmann correspondientes al caso en que G, Λ y c son functiones del tiempo, y en el caso particular en que $c(t)$ es una función lineal del tiempo – y manteniendo a G y Λ constantes –, resolvemos estas ecuaciones para obtener un estimado de la edad del universo.

ABSTRACT

We obtain the higher-order terms for the deflection of light around the black hole using the Lindstedt-Poincaré method to solve the equation of motion of a photon around a black hole. Additionally , we obtain diagonal Padé approximants from this perturbation expansion, and we show how these are a better fit for the numerical data. Furthermore, we show how these approximations can be used in ray-tracing algorithms to model the bending of light around a black hole. We also obtain the corresponding Friedmann equations for the case in which G, Λ and c are all functions of time, and in the case in which $c(t)$ is a linear function of time – and with constants G and Λ –, we solve the Friedmann equations to obtain an estimate of the age of the universe.

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Para los árboles que ayudan

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INTRODUCTION

The equations of General Relativity

General Relativity, perhaps one of the most elegant theories ever devised, was put forth by Albert Einstein in its current form in a 1916 publication, which expanded on his previous work of 1915. [1–3]. In these publications, Einstein showed both his famous fields equations, and the geodesic equations, which together form a total of 14 equations that govern General Relativity. Einstein's Field Equations can be written as:

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = kT_{\mu\nu} \tag{1.1}$$

where $G_{\mu\nu}$ is Einstein's tensor, which describes the curvature of space-time, Λ is the cosmological constant (originally introduced by Einstein in 1917 [4]), $g_{\mu\nu}$ is the metric tensor, $T_{\mu\nu}$ is the stress-energy tensor, which describes the content of matter and energy, and $k = \frac{8\pi G}{c^4}$, where c is the speed of light in vacuum and G is the gravitational constant. The Einstein's tensor is defined as follows:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{1.2}$$

where $R_{\mu\nu}$ is the Ricci tensor, and R is the Ricci scalar, which is the trace of the Ricci scalar

$$R = R_{\mu}{}^{\mu} \tag{1.3}$$

We can obtain an alternate form of Einstein's field equations by taking the trace of equation 1.1, and replacing this value on the original equation to obtain on equation for $R_{\mu\nu}$:

$$R_{\mu\nu} + \Lambda g_{\mu\nu} = k\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \tag{1.4}$$

where T is the trace of $T_{\mu\nu} : T = T^{\mu}{}_{\mu}$

Einstein's field equations describe how space-time curvature arises from the distribution of matter and energy. The equation of motion of a test particle inside a gravitational field is given by the geodesic equation:

$$\frac{dU^{\gamma}}{d\tau} + \Gamma^{\gamma}{}_{\alpha\beta} U^{\alpha} U^{\beta} = 0 \tag{1.5}$$

where a sum in repeated indices is implied, U^{μ} is the $\mu-$th component of the four-velocity vector of the particle, τ is the proper time of the particle, and $\Gamma^{\gamma}{}_{\alpha\beta}$ is the Christoffel symbol of the second kind, a measure of the dependence of the unit vectors with respect to the different coordinates [5].

Both of these equations were succinctly described by John Archibald Wheeler, referring respectively to the geodesic equation and the field equations: "Spacetime tells matter how to move; matter tells spacetime how to curve." [6]

Equation (1.1) is an equation between symmetric second-rank tensors, and in $4D$ space-time, it corresponds to a total of 10 highly non-linear differential equations for $g_{\mu\nu}$, the metric tensor, which corresponds to the deviation of Pythagoras's theorem in finding the dot product between two vectors with components A^{μ} and B^{μ}:

$$A \cdot B = A^{\mu} B_{\mu} = g_{\mu\nu} A^{\mu} B^{\nu} \tag{1.6}$$

$G_{\mu\nu}$ obeys the following property

$$G^{\mu\nu}_{;\nu} = 0 \tag{1.7}$$

where the semicolon on the indices indicate the covariant derivative of $G^{\mu\nu}$. Equation (1.1) implies that $T^{\mu\nu}{}_{;\nu} = 0$, and therefore, the $T_{\mu\nu}$ is related to a conserved quantity, which is a statement of the local conservation of energy.

Both equations (1.1) and (1.5) can be derived from an action principle. To derive Einstein's Field Equations, we can start from the following action:

$$S = \int_{\mathcal{R}} \mathrm{d}^4 x \sqrt{-g}\, [R - 2k\mathcal{L}_F + 2\Lambda] \tag{1.8}$$

where R, the Ricci scalar, one of the simplest non-trivial scalar related to the curvature [5], $\mathcal{L}_F$ is the Lagrangian density due to the fields of matter and energy, Λ is the cosmological constant, g is the determinant of the metric tensor, and we take the integral over all space-time. We need to take the variation of this action with respect to a variation of the metric, $\delta g_{\mu\nu}$ to obtain the Einstein's field equations. It is important to note that $\mathcal{L}$ is related to $T_{\mu\nu}$ as follows [7]:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \left[\frac{\partial\left(\sqrt{-g}\mathcal{L}\right)}{\partial g^{\mu\nu}} - \partial_\alpha \left(\frac{\partial\left(\sqrt{-g}\mathcal{L}\right)}{\partial(\partial_\alpha g^{\mu\nu})} \right) \right] \tag{1.9}$$

The following Lagrangian can be used to derive the geodesic equation for a test particle:

$$L\left(\frac{dx^\alpha}{d\sigma}, x^\alpha\right) = -g_{\alpha\beta}(x^\mu)\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma} \tag{1.10}$$

where σ is a parameter of the trajectory of the particle, which is usually taken to be the proper time, τ, or an affine parameter for massless particles [8]. An alternate form of the geodesic equation, easily derived from the previous Lagrangian, is the following:

$$\frac{dU_\alpha}{d\tau} = \frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha}U^\beta U^\gamma \tag{1.11}$$

This geodesic equation, with an affine parameter λ instead of the proper time τ (and $\frac{dx^\alpha}{d\lambda}$ instead of $U^\alpha = \frac{dx^\alpha}{d\tau}$) can be used to obtain the trajectory of massless particles, for which $\tau = 0$.

General Relativity and stellar bodies

In 1916, just a month after Einstein's 1915 paper, Karl Schwarzschild published forth the first non-trivial exact solution for the Einstein's Field Equations, which was corresponded to a spherical symmetric space-time, with a mass M in the center of the coordinate system [8]. Schwarzschild metric is the following, with coordinates $(x^0, x^1, x^2, x^3) = (ct, r, \theta, \phi)$:

$$g_{\mu\nu} = \begin{pmatrix} 1-\frac{r_s}{r} & 0 & 0 & 0 \\ 0 & -\left(1-\frac{r_s}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 sin^2\theta \end{pmatrix} \tag{1.12}$$

where r_s is the Schwarzschild radius:

$$r_s = \frac{2GM}{c^2} \tag{1.13}$$

For a radius $r < r_s$, all massless and massive test particles eventually reach $r = 0$. Thus, any particle (even photons) that falls beyond this Schwarzschild radius will not escape the black hole, neglecting quantum effects, such as Hawking Radiation [3].

General Relativity and Cosmology

Not long after Einstein developed his theory of General Relativity, he tested his theory on the whole Universe. However, in order to obtain a static universe, he added the cosmological constant, Λ, into his equations (1.1) to counteract the attractive forces of gravity [3]. However, observations by Slipher on the redshifts of galaxies were more similar to cosmological models by de Sitter's and Friedmann's cosmologies, the latter of which had a metric with an explicit dependence with time [4].

Current cosmologies theories depend on two main assumptions: the isotropy and homogeneity of the universe. These two assumptions lead to the Robertson-Walker metric [3], with coordinates $(x^0, x^1, x^2, x^3) = (ct, r, \theta, \phi)$:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{R^2(t)}{1-\hat{k}r^2} & 0 & 0 \\ 0 & 0 & -R^2(t)r^2 & 0 \\ 0 & 0 & 0 & -R^2(t)r^2 sin^2\theta \end{pmatrix} \tag{1.14}$$

where R(t) is a scale factor for the universe, and $\hat{k} = 0, -1, 1$ for a flat, open and closed Universe, respectively. From the Robertson-Walker metric, and employing equations (1.7) and (1.4), we can obtain both of Friedmann's equations:

$$\frac{1}{R}\frac{d^2R}{dt^2} = -\frac{kc^2}{6}(\rho + 3P) + \frac{\lambda c^2}{3} \tag{1.15}$$

$$H^2 = \left(\frac{1}{R}\frac{dR}{dt}\right)^2 = \frac{kc^2\rho}{3} - \frac{\hat{k}c^2}{R^2} + \frac{\lambda c^2}{3} \tag{1.16}$$

which can be used to calculate the age of the Universe, under assumptions of the curvature ($\hat{k}$ in the Robertson-Walker metric) and the composition of the universe [3].

Aim

The aim of the present work is to obtain the higher-order terms in the deflection of light around a massive object, and to test this values against numerical calculations. Additionally, we seek to obtain the equivalent to the Friedmann equations when G, c and Λ depend explicitly on time, and try to obtain the change in the age of the universe in the case of $c(t)$, for a speed of light that varies linearly with time.

DEFLECTION OF LIGHT AROUND A MASSIVE OBJECT

Geodesic equation for a photon in a Schwarzschild metric

Consider a photon traveling in the equatorial plane ($\theta = \pi/2$) around a black hole. For a photon (and other massless particles), $d\tau = 0$ and thus, we use an affine coordinate, λ, as a parameter for the trajectory instead of the proper time, τ. The geodesic equation, (1.11) for the coordinates ct (x^0) and ϕ (x^3) are:

$$\frac{d}{d\lambda}\left[\frac{d(ct)}{d\lambda}\left(1-\frac{r_s}{r}\right)\right]=0 \tag{2.1}$$

and

$$-\frac{d}{d\lambda}\left(\frac{d\phi}{d\lambda}r^2\right)=0 \tag{2.2}$$

Both of these equations define the following constants along the trajectory of the photon around the black hole:

$$c^2\frac{dt}{d\lambda}\left(1-\frac{r_s}{r}\right)=E \tag{2.3}$$

and

$$\frac{d\phi}{d\lambda}r^2=J \tag{2.4}$$

where E has units of energy per unit of mass and J has units of angular momentum per unit of mass (when λ has units of time). The invariant infinitesimal translation in the Schwarzschild metric for $\theta = \pi/2$ is given by:

$$c^2d\tau^2=c^2\left(1-\frac{r_s}{r}\right)dt^2-\left(1-\frac{r_s}{r}\right)^{-1}dr^2-r^2d\phi^2=0 \tag{2.5}$$

We can divide Equation (2.5) by $d\lambda^2$ on both sides, and use $\frac{dr}{d\lambda} = \frac{dr}{d\phi}\frac{d\phi}{d\lambda}$ to obtain:

$$c^2\left(1-\frac{r_s}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \left(1-\frac{r_s}{r}\right)^{-1}\left(\frac{dr}{d\phi}\right)^2\left(\frac{d\phi}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2 = 0 \tag{2.6}$$

If we multiply Equation (2.6) by $\left(1-\frac{r_s}{r}\right)$ in both sides, and insert the definitions of J and E we obtain:

$$\frac{E^2}{c^2} - \frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 J^2 - \frac{1}{r^2}\left(1-\frac{r_s}{r}\right)J^2 = 0 \tag{2.7}$$

This equation can be turned into an equation for $U(\phi) = \frac{1}{r(\phi)}$, noting that

$$\frac{dU}{d\phi} = -\frac{1}{r^2}\frac{dr}{d\phi} \tag{2.8}$$

so we arrive at the following equation for $U(\phi)$:

$$\frac{E^2}{c^2} - \left(\frac{dU}{d\phi}\right)^2 J^2 - U^2\left(1-r_sU\right)J^2 = 0 \tag{2.9}$$

By taking the derivative of Equation (2.9) with respect to ϕ, we arrive at the following differential equation for $U(\phi)$

$$\left(\frac{dU}{d\phi}\right)\left(2\frac{d^2U}{d\phi^2} + 2U - 3r_sU^2\right) = 0 \tag{2.10}$$

The differential equation in (2.10) can be separated into two differential equations for $U(\phi)$. The first one of this equation is the equation for a photon that travels directly into or out from the black hole:

$$\frac{dU}{d\phi} = 0 \tag{2.11}$$

the other differential equation, applicable for trajectories in which $U(\phi)$ is not constant with respect to ϕ, is the following:

$$\frac{d^2U}{d\phi^2} + U = \frac{3}{2}r_sU^2 \tag{2.12}$$

This equation can also be written in the following way, using the definition of the Schwarzschild radius:

$$\frac{d^2U}{d\phi^2} + U = \frac{3GMU^2}{c^2} \tag{2.13}$$

This is the equation for the trajectory of a massless particle that travels around a black hole in the equatorial plane.

Differential equation for the trajectory of a photon

In the previous section, we obtained the following differential equation for a photon traveling around a black hole:

$$\frac{d^2U}{d\phi^2} + U = \frac{3GMU^2}{c^2} \tag{2.14}$$

where the coordinates of the trajectory of the photon are given in spherical coordinates. The photon is taken to travel in the $\theta = 0$ plane, and in Equation (2.14), U is the reciprocal of the radial coordinate of the Schwarzschild metric , $U(\phi) = \frac{1}{r(\phi)}$. Equation (2.14) has an exact constant solution, for the unstable circular orbit of an electron around the black hole:

$$r_c = \frac{3GM}{c^2} \tag{2.15}$$

where r_c is the radius of the so-called photon sphere [9]. We note that the radius of the photon sphere can be expressed in terms of the Schwarzschild radius:

$$r_c = \frac{3r_s}{2} \tag{2.16}$$

The orbit described by a photon in the photon sphere is actually an unstable orbit , and a small perturbation in the orbit can lead either to the photon escaping the black hole or diving towards the event horizon [9].

Equation (2.14) is nonlinear, an highly difficult to solve analytically. However, a perturbative solution of this equation can be readily obtained, and is the basis for one of the experimental tests of General Relativity, the deflection of light around a massive object [9]. Let's first rewrite Equation (2.14) in terms of r_c:

$$\frac{d^2U}{d\phi^2} + U = r_c U^2 \tag{2.17}$$

Consider the initial conditions shown in Figure 2.1. The smallest value of the $r-$coordinate in the trajectory, $r = b$, is taken such that the photon escapes the black hole, $b > r_c$. Thus, let's try and rewrite Equation (2.17) in terms of $\epsilon = \frac{r_c}{b} < 1$, which we will use as a non-dimensional small number for our following perturbative expansions. Note that by multiplying both sides of Equation (2.17) by b, and defining the non-dimensional trajectory parameter

$$V(\phi) = \frac{b}{r(\phi)} \tag{2.18}$$

Equation (2.17), with the inclusion of the term $\epsilon = \frac{b}{r_c}$, then becomes a differential equation for equation in $V(\phi)$:

$$\frac{d^2V}{d\phi^2} + V = \epsilon V^2 \tag{2.19}$$

where $\epsilon = \frac{b}{r_c} < 1$, and with initial conditions given by

$$V(\phi = 0) = 1\,;\ \frac{dV}{d\phi}(\phi = 0) = 0 \tag{2.20}$$

In Appendix A we show that under these conditions, $V(\phi)$ is bounded such that

$$|V(\phi)| \leq 1 \tag{2.21}$$

First-order solution for $V(\phi)$

A first idea to obtain a solution of Equation (2.19) is to consider a $V(\phi)$ as a power series in ϵ:

$$V(\phi;\epsilon) = V_0(\phi) + \epsilon V_1(\phi) + \epsilon^2 V_2(\phi) + \ldots \tag{2.22}$$

Plugging the expansion (2.22) into Equation (2.19) results in the following:

$$\left(\frac{d^2V_0}{d\phi^2}+\epsilon\frac{d^2V_1}{d\phi^2}+\epsilon^2\frac{d^2V_2}{d\phi^2}+...\right)+(V_0+\epsilon V_1+\epsilon^2 V_2+...)=\epsilon\left(V_0+\epsilon V_1+\epsilon^2 V_2+...\right)^2 \quad (2.23)$$

We can group the powers of ϵ in Equation (2.23):

$$\epsilon^0 : \frac{d^2V_0}{d\phi^2}+V_0=0 \quad (2.24a)$$

$$\epsilon^1 : \frac{d^2V_1}{d\phi^2}+V_1={V_0}^2 \quad (2.24b)$$

$$\epsilon^2 : \frac{d^2V_2}{d\phi^2}+V_2=2V_0V_1 \quad (2.24c)$$

$$\epsilon^3 : \frac{d^2V_3}{d\phi^2}+V_3=(V_1)^2+2V_0V_2 \quad (2.24d)$$

$$\vdots$$

Note that the initial conditions of $V(\phi)$, applied to the asymptotic expansion in Equation (2.22), imply the following, by grouping powers of ϵ:

$$\epsilon^0 : V_0(0)=1\,;\ \frac{dV_0}{d\phi}(0)=0 \quad (2.25a)$$

$$\epsilon^k : V_k(0)=0\,;\ \frac{dV_k}{d\phi}(0)=0\,;\ k\geq 1 \quad (2.25b)$$

From these differential equations and initial conditions, we can readily obtain V_0 and V_1 iteratively [1]:

$$V_0(\phi)=cos(\phi) \quad (2.26)$$

$$V_1(\phi)=\frac{2}{3}-\frac{1}{3}cos(\phi)-\frac{1}{3}cos^2(\phi) \quad (2.27)$$

[1] It is convenient to write the $V_k(\phi)$ in terms of polynomials in $cos(\phi)$

Thus, we obtain an equation for $V(\phi)$, per Equation (2.22):

$$V(\phi) = cos(\phi) + \epsilon\left[\frac{2}{3} - \frac{1}{3}cos(\phi) - \frac{1}{3}cos^2(\phi)\right] + O(\epsilon^2) \tag{2.28}$$

Consider the truncated version of this equation:

$$\hat{V}(\phi) = \frac{2}{3}\epsilon + \left(1 - \frac{1}{3}\epsilon\right) cos(\phi) - \frac{1}{3}\epsilon\, cos^2(\phi) \tag{2.29}$$

According to the coordinate system shown in Figure 1, the photon goes through a total angular deflection of 2α. This corresponds to setting $V(\phi) = 0$ for both $\phi = \pi/2+\alpha$ and $\phi = -\pi/2 - \alpha$. From both of these conditions we get the same equation for $\alpha(\epsilon)$, assuming that $V(\phi)$ is a well-defined function of $cos(\phi)$. This occurs because

$$cos(\pi/2 + \alpha) = cos(-\pi/2 - \alpha) = -sin(\alpha) \tag{2.30}$$

Setting $V(\pi/2 + \alpha) = 0$ in Equation (2.29) gives the following equation for $sin(\alpha)$:

$$\epsilon sin^2\alpha + (3 - \epsilon)sin\alpha - 2\epsilon = 0 \tag{2.31}$$

Which is an equation for a second-order polynomial in $sin(\alpha)$. We can readily obtain $sin(\alpha)$ from Equation (2.31):

$$\sin(\alpha_\pm) = \frac{-3 + \epsilon \pm \sqrt{9 - 6\epsilon + 9\epsilon^2}}{2\epsilon} \tag{2.32}$$

The actual relevant solution for $sin(\alpha)$ is $sin(\alpha_+)$, which has a series expansion in ϵ of:

$$sin(\alpha_+) = \frac{2\epsilon}{3} + O(\epsilon^2) \tag{2.33}$$

whereas $sin(\alpha_-)$ has a series expansion with a leading term of order $1/\epsilon$:

$$sin(\alpha_-) = -\frac{3}{\epsilon} + 1 - \frac{2\epsilon}{3} + O(\epsilon^3) \tag{2.34}$$

Because both sides of Equation (2.33) are "small", $0 \leq \epsilon < 1$, we can approximate $sin(\alpha) \approx \alpha$ and obtain the first order deflection:

$$\alpha \approx \frac{2\epsilon}{3} \tag{2.35}$$

However, the more general method is to work with the inverse function of $sin(\alpha)$, employing the Taylor series of $arcsin(x)$ around $x = 0$. This approach will be used when obtaining α at higher orders. The total angular deflection of the light beam, $\Omega = 2\alpha$ is thus:

$$\Omega = \frac{4\epsilon}{3} + O(\epsilon^2) = \frac{4GM}{bc^2} + O\left[\left(\frac{GM}{bc^2}\right)^2\right] \tag{2.36}$$

Towards a second-order solution for $\Omega(\epsilon)$

We will now see how to obtain higher-order solutions for Ω. The differential equation in (2.24c) has the following solution:

$$V_2(\phi) = -\frac{4}{9} + \frac{41}{36}cos(\phi) + \frac{2}{9}cos^2(\phi) + \frac{1}{12}cos^3(\phi) + \frac{5}{12}\phi sin(\phi) \tag{2.37}$$

However, the term in Equation (2.37) that goes as $\phi sin(\phi)$ grows without bound, and occurs because the right-handed side of Equation (2.24c) contains terms proportional to the homogeneous solution of Equation (2.24c): $a\,cos(\phi) + b\,\sin(\phi)$. When this happens, the solution contains terms that grow without bound, such as $\phi sin(\phi)$, called *secular terms* [10]. Thus, if we naively include Equation (2.37) in $V(\phi)$, our solution is no longer bounded. Thus, we have to eliminate any and all secular term that arises to arrive at a well-behaved solution for $V(\phi)$.

One method to do this, due to Lindstedt and Poincaré, is by solving the differential equation in the following *strained coordinate* [10]:

$$\tilde{\phi} = \phi\left(1 + \omega_1\epsilon + \omega_2\epsilon^2 + \ldots\right) \tag{2.38}$$

Where the ω_k are constants to be determined.In terms of this new strained coordinate $\tilde{\phi}$, Equation (2.19) becomes

$$\left(1 + \omega_1\epsilon + \omega_2\epsilon^2 + \ldots\right)^2 \frac{d^2V}{d\tilde{\phi}^2} + V(\tilde{\phi}) = \epsilon V^2(\tilde{\phi}) \tag{2.39}$$

where we have used the chain rule for the derivative with respect to ϕ:

$$\frac{d}{d\phi} = \frac{d\tilde{\phi}}{d\phi}\frac{d}{d\tilde{\phi}} = \left(1 + \omega_1\epsilon + \omega_2\epsilon^2 + \ldots\right)\frac{d}{d\tilde{\phi}} \tag{2.40}$$

We proceed in the previous way, and assume an asymptotic expansion on $V(\tilde{\phi})$:

$$V(\tilde{\phi};\epsilon) = V_0(\tilde{\phi}) + \epsilon V_1(\tilde{\phi}) + \epsilon^2 V_2(\tilde{\phi}) + ... \tag{2.41}$$

Plugging in the expansion(2.41) in Equation (2.39), we obtain:

$$\begin{aligned}\left(1+\omega_1\epsilon+\omega_2\epsilon^2+\dots\right)^2\left(\frac{d^2V_0}{d\tilde{\phi}^2}+\epsilon\frac{d^2V_1}{d\tilde{\phi}^2}+\epsilon^2\frac{d^2V_2}{d\tilde{\phi}^2}+...\right)+\left(V_0+\epsilon V_1+\epsilon^2V_2+...\right)\\ =\epsilon\left(V_0+\epsilon V_1+\epsilon^2V_2+...\right)^2\end{aligned} \tag{2.42}$$

We can group the powers of ϵ in Equation (2.42):

$$\epsilon^0 : \frac{d^2V_0}{d\tilde{\phi}^2} + V_0 = 0 \tag{2.43a}$$

$$\epsilon^1 : \frac{d^2V_1}{d\tilde{\phi}^2} + V_1 = {V_0}^2 - 2\omega_1\frac{d^2V_0}{d\tilde{\phi}^2} \tag{2.43b}$$

$$\epsilon^2 : \frac{d^2V_2}{d\tilde{\phi}^2} + V_2 = 2V_0V_1 - ({\omega_1}^2 + 2\omega_2)\frac{d^2V_0}{d\tilde{\phi}^2} - 2\omega_1\frac{d^2V_1}{d\tilde{\phi}^2} \tag{2.43c}$$

$$\epsilon^3 : \frac{d^2V_3}{d\tilde{\phi}^2} + V_3 = {V_1}^2 + 2V_0V_2 - (2\omega_1\omega_2 + 2\omega_3)\frac{d^2V_0}{d\tilde{\phi}^2} - ({\omega_1}^2 + 2\omega_2)\frac{d^2V_1}{d\tilde{\phi}^2} - 2\omega_1\frac{d^2V_2}{d\tilde{\phi}^2} \tag{2.43c}$$

$$\vdots$$

With some care due to the definitions of the scaled variable and its derivative, we arrive at initial conditions for the $V_k(\tilde{\phi})$ from the initial conditions of $V(\phi)$:

$$\epsilon^0 : V_0(\tilde{\phi} = 0) = 1\,;\ \frac{dV_0}{d\tilde{\phi}}(0) = 0 \tag{2.44a}$$

$$\epsilon^k : V_k(\tilde{\phi} = 0) = 0\,;\ \frac{dV_k}{d\tilde{\phi}}(0) = 0\,;\ k \geq 1 \tag{2.44b}$$

Solving the differential equation (2.43a) with initial conditions (2.44a), we arrive at the zeroth-order contribution to $V(\tilde{\phi})$:

$$V_0(\tilde{\phi}) = cos(\tilde{\phi}) \tag{2.45}$$

Similarly, we can obtain $V_1(\tilde{\phi})$ from Equation (2.43b) subject to initial conditions (2.44b):

$$V_1(\tilde{\phi}) = \frac{2}{3} - \frac{1}{3}cos(\tilde{\phi}) - \frac{1}{3}cos^2(\tilde{\phi}) + \omega_1 \tilde{\phi} sin(\tilde{\phi}) \tag{2.46}$$

We note that a secular term has appeared for $V_1(\tilde{\phi})$. However, we use our freedom in the definition of ω_1 to eliminate this secular term by setting

$$\omega_1 = 0 \tag{2.47}$$

so that the final form of $V_1(\tilde{\phi})$ is:

$$V_1(\tilde{\phi}) = \frac{2}{3} - \frac{1}{3}cos(\tilde{\phi}) - \frac{1}{3}cos^2(\tilde{\phi}) \tag{2.48}$$

We proceed in a similar fashion and obtain $V_2(\tilde{\phi})$:

$$V_2(\tilde{\phi}) = -\frac{4}{9} + \frac{5}{36}cos(\tilde{\phi}) + \frac{2}{9}cos^2(\tilde{\phi}) + \frac{1}{12}cos^3(\tilde{\phi}) + \frac{1}{144}(144\omega_2 + 60)\tilde{\phi} sin(\tilde{\phi}) \tag{2.49}$$

To eliminate the secular term in $V_2(\tilde{\phi})$, we set

$$\omega_2 = -\frac{5}{12} \tag{2.50}$$

and obtain the well-behaved second-order term

$$V_2(\tilde{\phi}) = -\frac{4}{9} + \frac{5}{36}cos(\tilde{\phi}) + \frac{2}{9}cos^2(\tilde{\phi}) + \frac{1}{12}cos^3(\tilde{\phi}) \tag{2.51}$$

From all the solutions obtained so far, we can obtain the second-order correction to $\Omega(\epsilon)$. Note that $V(\tilde{\phi})$ is given by:

$$\begin{aligned} V(\tilde{\phi}) = cos(\tilde{\phi}) + \epsilon \left(\frac{2}{3} - \frac{1}{3}cos(\tilde{\phi}) - \frac{1}{3}cos^2(\tilde{\phi}) \right) \\ + \epsilon^2 \left(-\frac{4}{9} + \frac{5}{36}cos(\tilde{\phi}) + \frac{2}{9}cos^2(\tilde{\phi}) + \frac{1}{12}cos^3(\tilde{\phi}) + \right) + O(\epsilon^3) \end{aligned} \tag{2.52}$$

We set up $\tilde{\phi} = \pi/2 + \tilde{\alpha}$ in Equation (2.52), such that $V(\pi/2 + \tilde{\alpha}) = 0$ and obtain:

$$-sin(\tilde{\alpha}) + \epsilon\left(\frac{2}{3} + \frac{1}{3}sin(\tilde{\alpha}) - \frac{1}{3}sin^2(\tilde{\alpha})\right)$$
$$+\epsilon^2\left(-\frac{4}{9} - \frac{5}{36}sin(\tilde{\alpha}) + \frac{2}{9}sin^2(\tilde{\alpha}) - \frac{1}{12}sin^3(\tilde{\alpha}) +\right) + O(\epsilon^3) = 0 \tag{2.53}$$

We could truncate this Equation and solve the resultant cubic polynomial in $sin(\tilde{\alpha})$. However, this method would not be easy to generalize, because we do not have a general formula for the roots of fifth-order polynomials and above, according to Galois theory [11]. Also, an $n-$th order polynomial results in n different complex solutions, one of which we expect to have a leading term of order ϵ, to obtain a better approximation of Ω, and we would need to check all the n different solutions for this. Additionally, we have to remember that so far this is an asymptotic expansion in ϵ, and the truncation of the higher-order terms does not allow us to clearly see what the order of our estimate for $\Omega(\epsilon)$ is. All of these problems are solved by assuming that $sin(\tilde{\alpha})$ has the following expansion in ϵ, with a leading term of order ϵ^1:

$$sin(\tilde{\alpha}) = \epsilon\chi_1 + \epsilon^2\chi_2 + \epsilon^3\chi_3 + \dots \tag{2.54}$$

where the χ_k are constants to be determined. Inserting this new expansion into Equation (2.53) leads to the following algebraic Equation:

$$\left(\frac{2}{3} - \chi_1\right)\epsilon + \left(-\frac{4}{9} + \frac{\chi_1}{3} - \chi_2\right)\epsilon^2 + O(\epsilon^3) = 0 \tag{2.55}$$

We equate to zero the different powers of ϵ in this last Equation. Equating to zero the terms with ϵ^1 we arrive at:

$$\chi_1 = \frac{2}{3} \tag{2.56}$$

and equating to zero the terms with ϵ^2 we arrive at:

$$\chi_2 = -\frac{2}{9} \tag{2.57}$$

Thus, $sin(\tilde{\alpha})$ is given by:

$$sin(\tilde{\alpha}) = \frac{2}{3}\epsilon - \frac{2}{9}\epsilon^2 + O(\epsilon^3) \tag{2.58}$$

To obtain $\tilde{\alpha}$, we employ the Taylor series of $arcsin(x)$ around $x = 0$:

$$arcsin(x) = x + \frac{x^3}{6} + O(x^5) \tag{2.59}$$

and obtain

$$\tilde{\alpha} = \frac{2}{3}\epsilon - \frac{2}{9}\epsilon^2 + O(\epsilon^3) \tag{2.60}$$

However, what we actually want is α. From the definition of the strained coordinate $\tilde{\phi}$ in (2.38), it is clear that:

$$\frac{\pi}{2} + \alpha = \frac{\frac{\pi}{2} + \tilde{\alpha}}{1 + \omega_1\epsilon + \omega_2\epsilon^2 + O(\epsilon^3)} \tag{2.61}$$

From this latter Equation, an using the Taylor expansion of $\frac{1}{1+x}$ around $x = 0$, we obtain:

$$\alpha = \frac{2}{3}\epsilon + \left(\frac{5\pi}{24} - \frac{2}{9}\right)\epsilon^2 + O(\epsilon^3) \tag{2.62}$$

From which we can obtain the total deflection angle, $\Omega = 2\alpha$

$$\Omega = \frac{4}{3}\epsilon + \left(\frac{5\pi}{12} - \frac{4}{9}\right)\epsilon^2 + O(\epsilon^3) = \frac{4GM}{bc^2} + \left(\frac{15\pi}{4} - 4\right)\left(\frac{GM}{bc^2}\right)^2 + O\left[\left(\frac{GM}{bc^2}\right)^3\right] \tag{2.63}$$

Higher order solutions for $\Omega(\epsilon)$

The previous procedure can be automated to obtain high-order expressions for Ω. Notably, all the solutions for the $V_k(\tilde{\phi})$ are in the forms of $(k+1)-$order polynomials of $cos(\tilde{\phi})$, and a secular term that is eliminated by choosing a suitable ω_k. The use of the expansion of $\sin(\tilde{\alpha})$ in powers of ϵ guarantees both the form of $sin(\tilde{\alpha})$ with a leading term of order ϵ, and leads to algebraic equations for the χ_k that are exceedingly easy to solve. Notably, getting a higher-order solution conserves the lower-order terms. Consider the formal Taylor expansion of Ω around $\epsilon = 0$:

$$\Omega = \kappa_1\epsilon^1 + \kappa_2\epsilon^2 + \kappa_3\epsilon^3 + \ldots \tag{2.64}$$

A table of the coefficients κ_n of the series of Ω in (2.64) can be found in Table 2.1. These κ_n were found using the method of the previous sections, and obtaining the solutions up to $V_{20}(\tilde{\phi})$.

	Exact value	Numerical value
κ_1	$\frac{4}{3}$	1.33333
κ_2	$\frac{5\pi}{12} - \frac{4}{9}$	0.864552
κ_3	$\frac{122}{81} - \frac{5\pi}{18}$	0.633508
κ_4	$\frac{385\pi}{576} - \frac{130}{81}$	0.494911
κ_5	$\frac{7783}{2430} - \frac{385\pi}{432}$	0.403082
κ_6	$\frac{103565\pi}{62208} - \frac{21397}{4374}$	0.338319
κ_7	$\frac{544045}{61236} - \frac{85085\pi}{31104}$	0.290571
κ_8	$\frac{6551545\pi}{1327104} - \frac{133451}{8748}$	0.254143
κ_9	$\frac{1094345069}{39680928} - \frac{116991875\pi}{13436928}$	0.225577
κ_{10}	$\frac{2268110845\pi}{143327232} - \frac{1091492587}{22044960}$	0.202655
κ_{11}	$\frac{33880841953}{374134464} - \frac{18553890355\pi}{644972544}$	0.183902
κ_{12}	$\frac{3278312542505\pi}{61917364224} - \frac{627972527}{3779136}$	0.168300
κ_{13}	$\frac{17954674772417}{58364976384} - \frac{1514986498025\pi}{15479341056}$	0.155132
κ_{14}	$\frac{135335969751125\pi}{743008370688} - \frac{53937207017735}{94281884928}$	0.143875
κ_{15}	$\frac{1532445398265737}{1432594874880} - \frac{1138317723327785\pi}{3343537668096}$	0.134145
κ_{16}	$\frac{109432534129471767 5\pi}{1711891286065152} - \frac{4027582104301883}{2005632824832}$	0.125654
κ_{17}	$\frac{20646108759637948 27}{545532128354304} - \frac{128887453213429625\pi}{106993205379072}$	0.118179
κ_{18}	$\frac{1263396148548501892925\pi}{554652776685109248} - \frac{2657173119021192719}{371328591568896}$	0.111548
κ_{19}	$\frac{1085138496158025821251}{79959423384502272} - \frac{3993302456726670337 25\pi}{92442129447518208}$	0.105625
κ_{20}	$\frac{21869596358507403892 8865\pi}{2662333328088524390 4} - \frac{751868228052980757 61}{291350125692518 4}$	0.100303

TABLE 2.1: Coefficients κ_n of the series of Ω in (2.64). For these coefficients, we report both the exact values and the numerical values with 6 significant figures.

Clearly, as $\epsilon \to 1$, $\Omega(\epsilon) \to \infty$, because the photon starts going around the black hole as it starts closing in the photon sphere ($b \to r_c$). This means that $\Omega(\epsilon)$ has a singularity at $\epsilon = 1$. The Taylor expansion of $\Omega(\epsilon)$ around $\epsilon 0$ that we found at Equation (2.64) does not return an estimate for the position of this singularity, because a polynomial does not have a singularity. However, we can obtain Padé approximants for Ω around $\epsilon = 0$, and these will return an estimate for the position of this singularity.

As a small refresher on Padé approximants, we note their definition. A Padé approximant of a function $f(x)$ is a rational function $f^{[L|M]}(x)$ of the form:

$$f^{[L|M]}(x) = \frac{a_0 + a_1 x + a_2 x^2 + ... + a_L x^L}{1 + b_1 x + b_2 x^2 + ... + b_M x^M} \tag{2.65}$$

where $f(x)$ and $f^{[L|M]}(x)$ are equal in their first $L + M + 1$ derivatives around $x = 0$ [12, 13]. A diagonal Padé approximant $f^{[N]}(x)$ is a Padé approximant in which $N = L = M$. We can obtain the diagonal Padé approximants for up to N=10 with the Taylor series expansion for $\Omega(\epsilon)$. For example, the $\Omega^{[1]}(\epsilon)$ Padé approximant is given by:

$$\Omega^{[1]}(\epsilon) = \frac{48\pi + (64 + 16\pi - 15\pi^2)\,\epsilon}{96 + (32 - 30\pi)\epsilon} \tag{2.66}$$

The exact formulas for the Padé approximants of $\Omega(\epsilon)$ are rather complicated because of the powers of π involved. Due to this, our work with Padé approximants will be purely numeric. All the Padé approximants $\Omega^{[N]}(\epsilon)$ have a singularity of order 1 at a position around $\epsilon = 1$. The position of this singularity, ϵ_s, is tabulated for the 10 Padé approximants in 2.2.

N	ϵ_s
1	1.54222
2	1.21736
3	1.11036
4	1.06664
5	1.04532
6	1.03238
7	1.0245
8	1.01915
9	1.01537
10	1.01264

TABLE 2.2: The position of the singularity near $\epsilon = 1$ for the Padé approximants, $\Omega^{[N]}(\epsilon)$.

Numerical tests for $\Omega(\lambda)$ and its Padé approximants

All the coefficients for the Taylor expansion of $\Omega(\epsilon)$ were obtained around $\epsilon = 0$. We can test the correctness of the methods thus far used to obtain this function by comparing it to the results of numerical solutions of Equation (2.14). This is done with both truncated $n-$th order Taylor polynomials from $\Omega(\epsilon)$ and for the Padé approximants we obtain from this function, $\Omega^{[N]}(\epsilon)$. This comparisons are shown in Figures 2.2 and 2.3.

We see from Figures 2.2 and 2.3 that the Padé approximants are much faster at converging into the actual form of $\Omega(\epsilon)$. The convergence of the Padé approximants is such that for $\epsilon = 0.99$, the $N = 10$ diagonal Padé approximant is within 3% of the corresponding numerical value. This is mainly due to the fact that Padé approximants are better in approximating functions that have singularities [13]. Once we know that the $\Omega(\epsilon)$ behave correctly, we can use the $\Omega(\epsilon)$ to simulate the bending of light around a black hole. A simple first–order ray tracing algorithm that does this for the different approximations of $\Omega(\epsilon)$ we have found is shown in Appendix B.

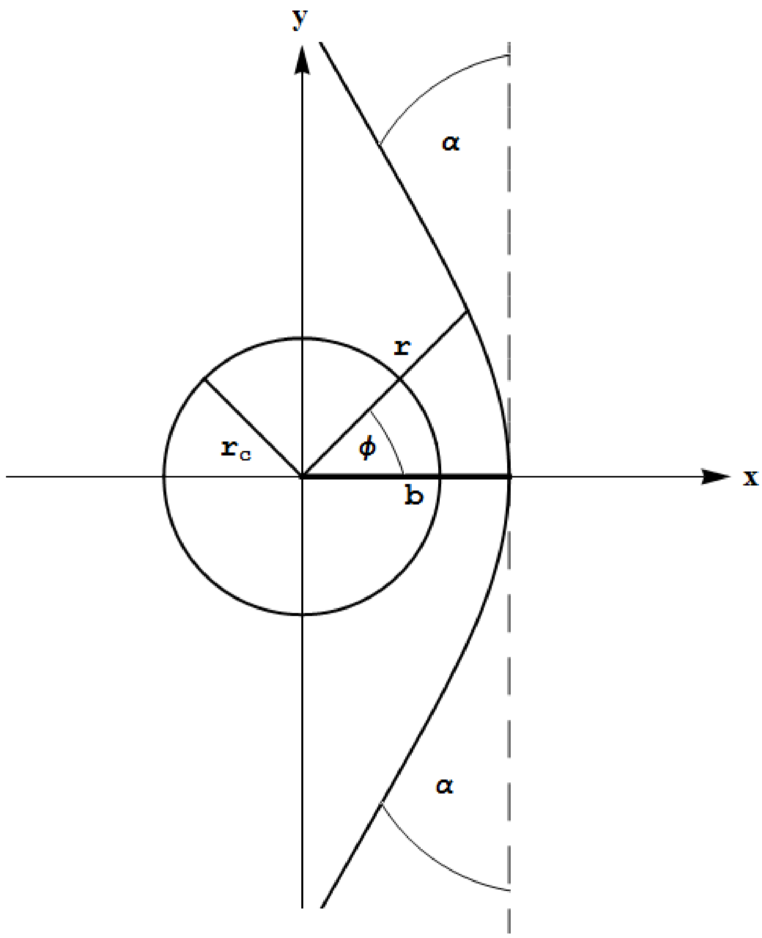

FIGURE 2.1: Trajectory of a photon outside the photon sphere. The initial conditions are taken such that $r|_{\phi=0} = b$, and is called the impact parameter of the trajectory - the closest distance from the trajectory to the center of the black hole. We thus have, $\frac{dr}{d\phi}|_{\phi=0} = 0$ and $\frac{dU}{d\phi}|_{\phi=0} = 0$. The photon experiences a total angular deflection of 2α.

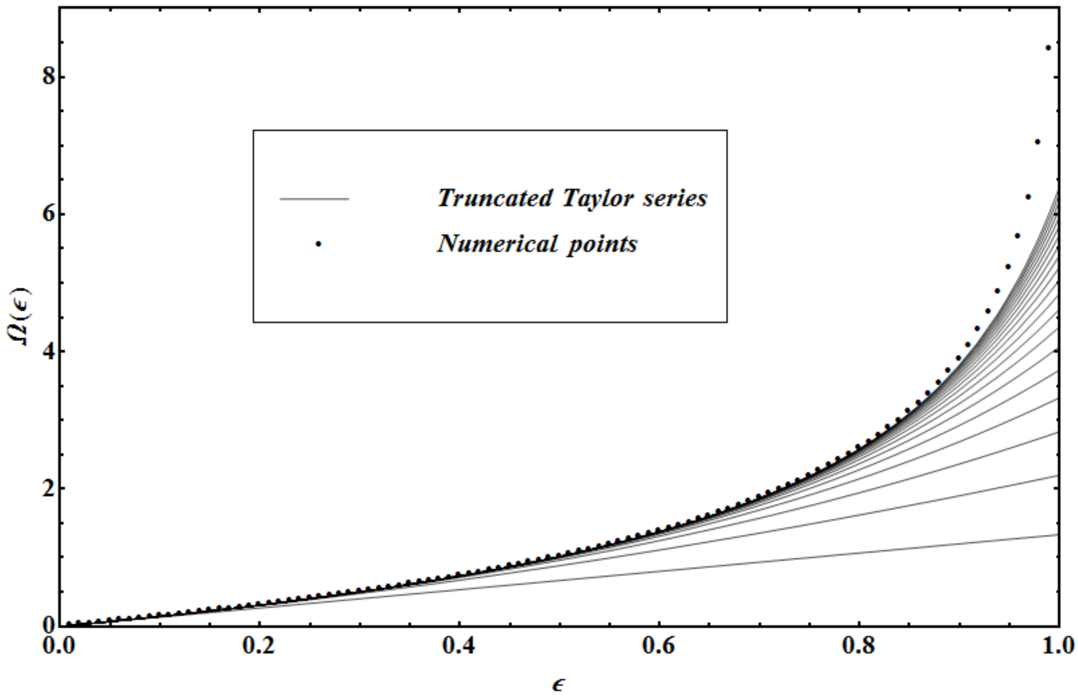

FIGURE 2.2: Numerical points obtained for $\Omega(\epsilon)$ compared to the truncated $n-$th order Taylor polynomials of $\Omega(\epsilon)$, up to $20-$th order. With increasing value of n, the polynomials take larger values.

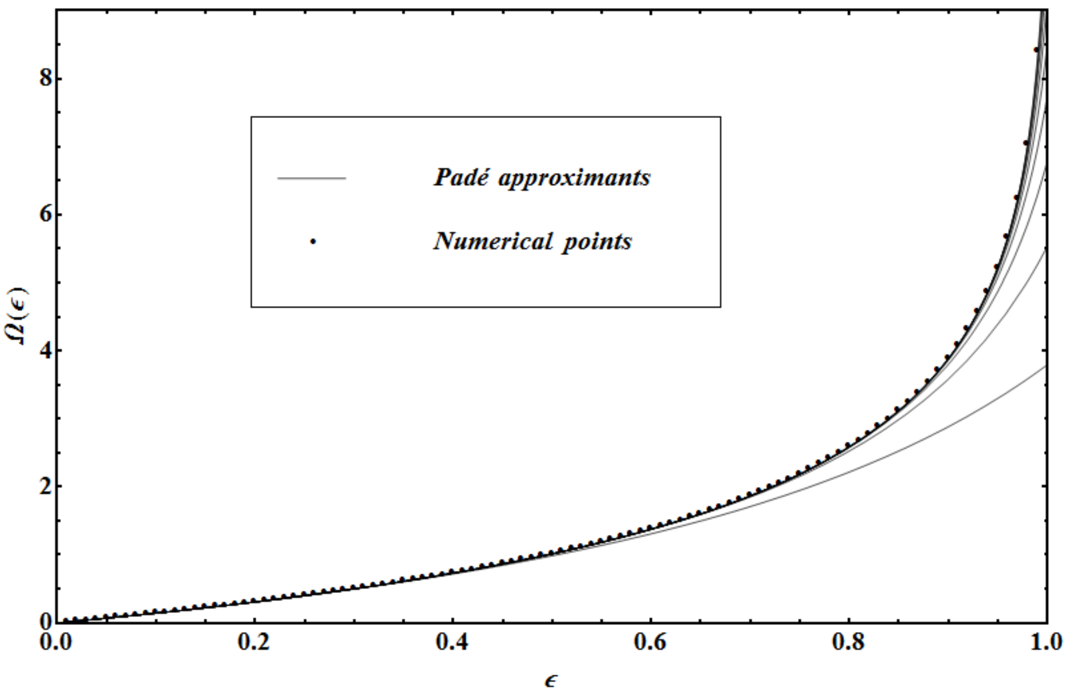

FIGURE 2.3: Numerical points obtained for $\Omega(\epsilon)$ compared to the truncated $N-$th diagonal Padé approximants of $\Omega(\epsilon)$, up to $N = 10$. With increasing value of N, the Padé approximants take larger values. For $\epsilon = 0.99$, the $N = 10$ Padé approximant is within 3% of the numerical value of $\Omega(\epsilon)$.

Friedmann equations for time dependent G, Λ, and c

Einstein field equations with time-dependent G, Λ, c

We will now consider G, Λ and c to be all function of time: $G(t)$, $\Lambda(t)$, and $c(t)$. Doing this for the first two quantities removes some of the covariance of our equations, making the time-coordinate $x^0 = c(t)t$ explicitly different from the rest. We also clarify now what we mean by saying that the speed of light, c, varies with respect to time: at different cosmological times t, the conversion factor between what we call time and the coordinate x^0, which is just the speed of light c, varies with time. Thus, in the formulae that require derivatives with respect to the coordinate x^0, we shall use the following:

$$\frac{\partial}{\partial x^0} = \frac{\partial}{\partial (c(t)t)} = \frac{1}{c + t\frac{\mathrm{d}c}{\mathrm{d}t}} \frac{\partial}{\partial t} \tag{3.1}$$

Under these conditions, the Hilbert action is given by the integral over all space-time:

$$S = \int_{\mathcal{R}} \mathrm{d}^4 x \left[R - 2k(t)\mathcal{L}_F + 2\Lambda(t) \right] \tag{3.2}$$

where $k(t) = \frac{8\pi G(t)}{c^4(t)}$. Because the variation of this action is taken with respect to a variation of the metric, $\delta g^{\mu\nu}$, and noting that neither $k(t)$ nor $\Lambda(t)$ depend explicitly on the metric coefficients nor its derivatives – note Equation (1.9)–, we obtain a similar expression for Einstein's field equations:

$$G_{\mu\nu} - \Lambda(t) g_{\mu\nu} = k(t) T_{\mu\nu} \tag{3.3}$$

which we can put in a form similar to the form of Equation (1.4):

$$R_{\mu\nu} + \Lambda(t) g_{\mu\nu} = k(t) \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \tag{3.4}$$

Also note that

Because the Robertson-Walker metric is obtained from purely geometrical means (considerations of homogeneity and isotropy of the universe), we posit that the Robertson-Walker metric is also a solution to Equation (3.4). The next step should then be to derive equations equivalent to the Friedmann equations, using the Robertson-Walker metric, and Equations (3.3), (3.4), and (1.7).

Form of the Friedmann equations with $G(t)$, $\Lambda(t)$, and $c(t)$

At large scales, we can consider the universe to be made of a perfect fluid of energy density ρ and pressure p, and moving with 4-velocity U^μ. This perfect fluid would have the following energy-momentum tensor [3]:

$$T_{\mu\nu} = \frac{\rho + p}{c^2} U_\mu U_\nu - p g_{\mu\nu} \tag{3.5}$$

And in a reference frame that is in rest with respect to the universal fluid, $U^\mu = (c, 0, 0, 0)$. With this, the mixed-index energy-momentum tensor is given by[1]:

$$T_\mu^\nu = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix} \tag{3.6}$$

Equation (1.7), lowering the μ index, implies that:

$$(k(t) T_\mu{}^\nu + \Lambda(t) \delta_\mu{}^\nu)_{;\nu} = 0 \tag{3.7}$$

this Equation can be simplified into

$$k_{;\nu} T_\mu{}^\nu + k T_\mu{}^\nu{}_{;\nu} + \Lambda(t)_{;\nu} \delta_\mu{}^\nu = 0 \tag{3.8}$$

noting that $\delta_\mu{}^\nu$ is constant. We are actually interested in solely the $\mu = 0$ component of Equation (3.8). The most complicated term we have to evaluate for the $\mu = 0$ component of Equation (3.8) is

$$T_0{}^\nu{}_{;\nu} = \frac{\partial}{\partial x^\nu} T_0{}^\nu - T_\sigma{}^\nu \Gamma^\sigma{}_{0\nu} + T_0{}^\sigma \Gamma^\nu{}_{\sigma\nu} \tag{3.9}$$

[1]note that we use $g^\mu{}_\nu = \delta^\mu{}_\nu$

The only Christoffel symbols different from zero in Equation (3.9) are

$$\Gamma^1{}_{01} = \Gamma^2{}_{02} = \Gamma^2{}_{03} = \frac{\frac{\partial}{\partial x^0} R(t)}{R(t)} \tag{3.10}$$

With the Christoffel symbols in (3.10) and with the use of Equation (3.9), the $\mu = 0$ component of Equation (3.7) becomes

$$\rho \frac{\partial k}{\partial x^0} + k \left[\frac{\partial \rho}{\partial x^0} + 3 \frac{\frac{\partial}{\partial x^0} R(t)}{R(t)} (\rho + p) \right] + \frac{\partial \Lambda}{\partial x^0} = 0 \tag{3.11}$$

Equation (3.11) is equivalent to the conservation of energy equation [3]. Due to the nature in which x^0 is defined, we can readily obtain the components of the Ricci tensor from reference [3]. In this reference, professor Carroll uses a $(-1, 1, 1, 1)$ sign convention, but we note that the Ricci tensor is invariant under the change of sign convention, $g_{\mu\nu} \to -g_{\mu\nu}$, because the Christoffel symbols are in turn unchanged by a change in the sign of the metric. Thus, we can read both the Christoffel symbols and the component of the Ricci tensor directly from Equations (8.44) and (8.45) of reference [3]. With this in mind, we write the following components of the Ricci tensor, which is diagonal [3]:

$$R_{00} = -3 \frac{\frac{\partial^2}{\partial x^{0^2}} R(t)}{R(t)} \tag{3.12}$$

$$R_{22} = R^2(t) \left[R(t) \frac{\partial^2}{\partial x^{0^2}} R(t) + 2 \left(\frac{\partial}{\partial x^0} R(t) \right)^2 + 2\hat{k} \right] \tag{3.13}$$

The $00-$component of Equation 3.4, using expression (3.12), is:

$$-3 \frac{\frac{\partial^2}{\partial x^{0^2}} R(t)}{R(t)} + \Lambda(t) = \frac{k(t)}{2} (\rho + 3p) \tag{3.14}$$

where we used that $T = \rho - 3p$. This expression can be further simplified as:

$$\frac{\frac{\partial^2}{\partial x^{0^2}} R(t)}{R(t)} = -\frac{k}{6} (\rho + 3p) + \frac{\Lambda(t)}{3} \tag{3.15}$$

which is equivalent to Equation (1.15), the Friedman acceleration Equation. The $22-$component of Equation 3.4, using expression (3.13), is:

$$R^2(t)\left[R(t)\frac{\partial^2}{\partial x^{0^2}}R(t)+2\left(\frac{\partial}{\partial x^0}R(t)\right)^2+2\hat{k}\right]-\Lambda(t)R^2(t)r^2=-k(t)r^2R^2(t)(p-\rho) \quad (3.16)$$

this latter equation can be further simplified into:

$$\frac{\frac{\partial^2}{\partial x^{0^2}}R(t)}{R(t)}+2\left(\frac{1}{R(t)}\frac{\partial R}{\partial x^0}\right)^2+\frac{\hat{k}}{R^2}-\Lambda(t)=-\frac{k(t)}{2}(p-\rho) \quad (3.17)$$

The value of $\frac{\frac{\partial^2}{\partial x^{0^2}}R(t)}{R(t)}$ in Equation (3.15) can be inserted in Equation (3.17) to obtain, after some simplifications:

$$\left(\frac{1}{R(t)}\frac{\partial R}{\partial x^0}\right)^2=\frac{k(t)\rho}{3}-\frac{\hat{k}}{R^2(t)}+\frac{\Lambda(t)}{3} \quad (3.18)$$

which is equivalent to the Friedmann Equation in (1.16). Equations (3.11), (3.15), and (3.18) are the equations of motion of this cosmology.

It is convenient to add the cosmological constant Λ into the energy density in Equation (3.18), using

$$\frac{\Lambda}{3}=\frac{k\rho_\Lambda}{3} \quad (3.19)$$

we obtain the following Equation:

$$\left(\frac{1}{R(t)}\frac{\partial R}{\partial x^0}\right)^2=\frac{k(t)\rho}{3}-\frac{\hat{k}}{R^2(t)} \quad (3.20)$$

where ρ now includes the contributions due to Λ. In the next section, we will consider a solution to these equations for a time-dependent speed of light, $c(t)$, with the other two parameters held constant.

Age of the universe with a speed of light that varies linearly with time

We will consider one of the simplest form of time-dependence: that of constant $\frac{dc}{dt}$. We will now use Equation(3.20) with constants $k(t)$ and $\Lambda(t)$:

$$\left(\frac{1}{R(t)}\frac{\partial R}{\partial x^0}\right)^2 = \frac{k\rho}{3} - \frac{\hat{k}}{R^2(t)} \tag{3.21}$$

It is convenient to define the critical density, ρ_c:

$$\rho_c = \frac{3}{k}\left(\frac{1}{R(t)}\frac{\partial R}{\partial x^0}\right)^2 \tag{3.22}$$

and the density parameter, Ω:

$$\Omega = \frac{\rho}{\rho_c} \tag{3.23}$$

noting that both of these parameters are time-dependent. With these two parameters in mind, Equation (3.21) can be written as

$$\frac{\hat{k}}{R^2} = \left(\frac{1}{R(t)}\frac{\partial R}{\partial x^0}\right)^2 (\Omega - 1) \tag{3.24}$$

Another important parameter to consider is a parameter that takes the time-dependence of $R(t)$, $a(t)$:

$$R(t) = R_0 a(t) \tag{3.25}$$

such that at our present time, t_0, $a(t_0) = 1$. With this parameter, we can now write:

$$\frac{\hat{k}}{R_0^2} = \left(\frac{1}{R(t)}\frac{\partial R}{\partial x^0}\right)_0^2 (\Omega_0 - 1) \tag{3.26}$$

where the 0 index refers to the parameters evaluated at the present time. If we insert back this last Equation into Equation (3.21), and divide by $\left(\frac{1}{R(t)}\frac{dR}{dx^0}\right)_0^2$ in both sides, we obtain:

$$\left(\frac{1}{R(t)}\frac{\partial R}{\partial x^0}\right)_0^{-2}\left(\frac{1}{R(t)}\frac{\partial R}{\partial x^0}\right)^2 = \frac{\rho}{\rho_{c,0}} + (1 - \Omega_0)\frac{1}{a^2(t)} \tag{3.27}$$

Equation (3.11), with Λ and k constants, and using the relation $p = \omega\rho$, implies that [3]:

$$\rho \propto R^{-3(\omega+1)}(t) \propto a^{-3(\omega+1)}(t) \tag{3.28}$$

where we have different vales for ω for matter, radiation and vacuum energy (Λ): $\omega = 0, 1/3, -1$, respectively. From this, and from defining piecewise density parameters, such as $\Omega_\Lambda = \rho_\Lambda/\rho_c$, we arrive at the following differential Equation:

$$\left(\frac{1}{R(t)}\frac{\partial R}{\partial x^0}\right)_0^{-2}\left(\frac{1}{R(t)}\frac{\partial R}{\partial x^0}\right)^2 = \left[\frac{\Omega_{r,0}}{a^4(t)} + \frac{\Omega_{m,0}}{a^3(t)} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2(t)}\right] \tag{3.29}$$

where $\Omega_{m,0}$ is the density parameter for matter (and dark matter) at the current time, $\Omega_{r,0}$ is the density parameter for radiation at the current time and Ω_Λ is the energy parameter associated to dark energy (i.e. due to the Cosmological constant). This equation can be put into a more readable form by using the Hubble parameter, $H(t)$ (note: $R(t)$ is a function of time alone, so partial derivatives become total derivatives in this definition):

$$H(t) = \frac{1}{R(t)}\frac{dR}{dt} \tag{3.30}$$

and by using the derivative rule of Equation (3.1), we obtain [2]:

$$\frac{\left(\frac{c_0}{c+t\frac{dc}{dt}}\right)^2}{\left(\frac{c}{c+t\frac{dc}{dt}}\right)_0^2}\frac{H^2}{H_0^2} = \left[\frac{\Omega_{r,0}}{a^4(t)} + \frac{\Omega_{m,0}}{a^3(t)} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2(t)}\right] \tag{3.31}$$

There is one last detail to consider before we try to solve this equation. We have to note that astronomers measure the Ω without the factors that appear in (3.1) due to the time-dependence of $c(t)$ in $\frac{1}{R}\frac{dR}{dx^0}$. For any $k = m, r, \Lambda$ and for the total density parameter, we defined:

$$\Omega_{k,0} = \frac{\rho_{k,0}}{\rho_{c,0}} = \frac{\rho_{k,0}}{\frac{3}{k}\left(\frac{1}{R(t)}\frac{\partial R}{\partial x^0}\right)_0^2} = \frac{\rho_{k,0}}{\frac{3H_0^2}{kc_0^2}\left(\frac{c}{c+t\frac{dc}{dt}}\right)_0^2} \tag{3.32}$$

[2]we multiply the numerator and the denominator in the left side by c_0^2, the speed of light at the present time, so these become non-dimensional numbers

where c_0 is the current value of the speed of light. Meanwhile an astronomer would use Equation (3.22), taking c to be a constant, to obtain:

$$\rho_{c,0}^{\text{astro}} = \frac{3H_0^2}{kc_0^2} \tag{3.33}$$

and thus would make use of the following definition:

$$\Omega_{k,0}^{\text{astro}} = \frac{\rho_{k,0}}{\rho_{c,0}^{\text{astro}}} = \frac{\rho_{k,0}}{\frac{3H_0^2}{kc_0^2}} \tag{3.34}$$

Thus, to make use of the values of the density parameters as measured by astronomers, we have to use the following substitution:

$$\Omega_{k,0} = \frac{\Omega_{k,0}^{\text{astro}}}{\left(\frac{c}{c+t\frac{dc}{dt}}\right)_0^2} \tag{3.35}$$

Making use of Equation (3.35) in Equation (3.31), and multiplying both sides by $\left(\frac{c}{c+t\frac{dc}{dt}}\right)_0^2$, we obtain

$$\left(\frac{c_0}{c+t\frac{dc}{dt}}\right)^2 \frac{H^2}{H_0^2} = \left[\frac{\Omega_{r,0}^{\text{astro}}}{a^4(t)} + \frac{\Omega_{m,0}^{\text{astro}}}{a^3(t)} + \Omega_{\Lambda,0}^{\text{astro}} + \frac{\left(\frac{c}{c+t\frac{dc}{dt}}\right)_0^2 - \Omega_0^{\text{astro}}}{a^2(t)}\right] \tag{3.36}$$

This can be simplified further, as long as

$$\left| \frac{t_0}{c_0}\frac{dc}{dt}\bigg|_{t_0} \right| << 1 \tag{3.37}$$

we can write

$$\left(\frac{c}{c+t\frac{dc}{dt}}\right)_0^2 = 1 - \delta(t_0) \tag{3.38}$$

where $\delta(t_0)$ is given by

$$\delta(t_0) = 2\frac{t_0}{c_0}\frac{dc}{dt}\bigg|_{t_0} + O\left[\left(\frac{t_0}{c_0}\frac{dc}{dt}\bigg|_{t_0}\right)^2\right] \tag{3.39}$$

With the definition of $\delta(t_0)$, Equation (3.31) becomes

$$\left(\frac{c_0}{c+t\frac{dc}{dt}}\right)^2\frac{H^2}{H_0^2}=\left[\frac{\Omega_{r,0}^{\text{astro}}}{a^4(t)}+\frac{\Omega_{m,0}^{\text{astro}}}{a^3(t)}+\Omega_{\Lambda,0}^{\text{astro}}+\frac{1-\Omega_0^{\text{astro}}-\delta(t_0)}{a^2(t)}\right] \tag{3.40}$$

Noting that

$$H=\frac{1}{a}\frac{da}{dt} \tag{3.41}$$

Equation (3.40) can be separated as follows[3]:

$$H_0\int_0^{t_0}\mathrm{d}t\frac{c+t\frac{dc}{dt}}{c_0}=\int_0^1\mathrm{d}a\frac{1}{a}\left[\frac{\Omega_{r,0}^{\text{astro}}}{a^4(t)}+\frac{\Omega_{m,0}^{\text{astro}}}{a^3(t)}+\Omega_{\Lambda}^{\text{astro}}+\frac{1-\Omega_0^{\text{astro}}-\delta(t_0)}{a^2(t)}\right]^{-1/2} \tag{3.42}$$

We can define

$$T(\Omega_0,\Omega_{r,0},\Omega_{m,0},\Omega_{\Lambda,0})=\frac{1}{H_0}\int_0^1\frac{\mathrm{d}a}{\left[\frac{\Omega_{r,0}}{a^4(t)}+\frac{\Omega_{m,0}}{a^3(t)}+\Omega_{\Lambda,0}+\frac{1-\Omega_0}{a^2(t)}\right]^{1/2}} \tag{3.43}$$

where $T(\Omega_0,\Omega_{r,0},\Omega_{m,0},\Omega_{\Lambda,0})$ is easy to interpret: it would be the age of the Universe employing this cosmological model if we did not take into account the time-dependence of $c(t)$. The integrand in the left-hand side of Equation (3.42) is:

$$\frac{c+t\frac{dc}{dt}}{c_0}=\frac{c_0+(2t-t_0)\frac{dc}{dt}}{c_0} \tag{3.44}$$

where we have explicitly put the linear time-dependence of $c(t)$, $c(t)=c_0+(t-t_0)\frac{dc}{dt}$. By integrating Equation (3.42) we obtain:

$$t_0=T(\Omega_0^{\text{astro}}+\delta(t_0),\Omega_{r,0}^{\text{astro}},\Omega_{m,0}^{\text{astro}},\Omega_{\Lambda,0}^{\text{astro}}) \tag{3.45}$$

Equation (3.45) can be solved iteratively until we obtain a converging value for t_0. As a first iteration, if we set $\delta(t_0)=0$, we would arrive at the usual age of the Universe, $t_0=13.813\pm0.038$ Gyr [14]. Let's now consider an experimental bound for $\frac{1}{c}\frac{dc}{dt}$ in order

[3]This equation is valid for any $c(t)$ in which Equation (3.37) holds.

to obtain a second iteration for t_0. We will use the following experimental bound for the fine-structure constant, $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$ [15], with 1 significant figure:

$$\frac{1}{\alpha}\frac{d\alpha}{dt} = 2\times 10^{-16}\ \text{yr}^{-1} \tag{3.46}$$

From the definition of the fine-structure constant, it is straightforward to show that

$$\frac{d\alpha}{\alpha} = -\frac{dc}{c} \tag{3.47}$$

assuming that the changes in α are purely due to changes in c. Thus, the experimental bound for $\frac{1}{c_0}\frac{dc}{dt} = -2\times 10^{-16}\ \text{yr}^{-1}$. With this value, we obtain for our second iteration:

$$\delta(t_0) = -6\times 10^{-6} \tag{3.48}$$

This value is much smaller than the uncertainty in both $\Omega^{\text{astro}}_{m,0}$ and $\Omega^{\text{astro}}_{\Lambda,0}$, which are both ± 0.0062 in the most precise estimates by Planck experiment [14]. Clearly, any further iteration for t_0 will return a value well within the experimental and statistical uncertainties for the age of the universe. Therefore, no more iterations will be done for t_0.

Conclusions and Recommendations

We have successfully obtained both expressions for $\Omega(\epsilon)$, the angular deflection of a photon traveling around a black hole, and $\delta(t_0)$, the change in the overall density parameter that arises from considering a speed of light that varies linearly with time. In the first case, we assumed that the parameter ϵ was small, and we were able to obtain $\Omega(\epsilon)$ at such a high order in ϵ (and with the use of Padé approximants) that the best approximation for $\Omega(\epsilon)$ we obtained was consistent with the numerical data even for an $\epsilon \approx 0.99$. Furthermore, we were able to use this estimate for $\Omega(\epsilon)$ to simulate the bending of light around a black hole.

Interestingly, when we were trying to find how the lifetime of the Universe changes when we consider a time-dependent speed of light, the actual effect of this was mainly observed in the value of Ω_0, the present-day value for the overall density parameter. We found out that this varies, at first order, by $\delta(t_0) = 6 \times 10^{-6}$. This change of Ω_0, however, does not result into a change of the lifetime of the Universe above its current uncertainty, because the uncertainty in both $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ are two orders of magnitude higher than $\delta(t_0)$. We also arrived at a general result for t_0 where the speed of light is a function of time, Equation (3.42):

$$H_0 \int_0^{t_0} \mathrm{d}t \frac{c + t\frac{dc}{dt}}{c_0} = \int_0^1 \mathrm{d}a \frac{1}{a} \left[\frac{\Omega_{r,0}^{\text{astro}}}{a^4(t)} + \frac{\Omega_{m,0}^{\text{astro}}}{a^3(t)} + \Omega_{\Lambda}^{\text{astro}} + \frac{1 - \Omega_0^{\text{astro}} - \delta(t_0)}{a^2(t)} \right]^{-1/2}$$

In order to arrive at this equation, the only assumption we employed was $\left(\frac{t_0}{c}\frac{dc}{dt}\right)_0 << 1$.

Still, there are some things that have been left undeveloped in the current work. For example, we did not try to solve either the Einstein Field Equations (for a metric different than the Robertson-Walker metric), nor the Friedmann equations where G and Λ where functions of time. Likewise, we did not analyze what differences, if any, could be obtained with our approximation of $\Omega(\epsilon)$ when using gravitational lensing to estimate amounts of mass in galaxy clusters. Equation (3.42) could also be employed for models of $c(t)$ other than the linearly–varying model we studied in this work. These might well be topics of interest for future colleagues.

ON THE BOUNDEDNESS OF V(θ)

Consider the differential Equation for $V(\phi)$, Equation (2.19):

$$\frac{d^2V}{d\phi^2} + V = \epsilon V^2 \tag{A.1}$$

subject to initial the initial conditions

$$V(0) = 1\,;\; \frac{dV}{d\phi}(0) = 0 \tag{A.2}$$

and where $\epsilon < 1$. Equation (A.1) is equivalent to the 1-D equation of motion of a mass ($m = 1$) on a spring (with spring constant $k = 1$) with an additional higher-order term:

$$\frac{d^2x}{dt^2} + x(t) = \epsilon\, x^2(t) \tag{A.3}$$

with equivalent initial conditions:

$$x(0) = 1\,;\; \frac{dx}{dt}(0) = 0 \tag{A.4}$$

Consider the following 1-D potential energy function:

$$E_P(x) = \frac{1}{2}x^2 - \frac{1}{3}\epsilon x^3 \tag{A.5}$$

It is straightforward to show that a unit mass ($m = 1$) affected by this potential follows the equation of motion in (A.3), noting that

$$F = \frac{d^2x}{dt^2} = -\frac{dE_P}{dx} \tag{A.6}$$

More importantly, this means that the total energy of the unit mass is a constant of motion, given by the initial conditions in (A.4):

$$E_{tot} = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}x^2 - \frac{1}{3}\epsilon x^3 = \frac{1}{2} - \frac{1}{3}\epsilon \tag{A.7}$$

and noting that kinetic energy is non-negative:

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 \geq 0 \tag{A.8}$$

Therefore, the unit mass particle in the potential well given by E_P can only have positions x(t) whenever

$$E_P(x) = \frac{1}{2}x^2 - \frac{1}{3}\epsilon x^3 \leq E_{tot} = \frac{1}{2} - \frac{1}{3}\epsilon \tag{A.9}$$

and the trajectory of the unit mass is given by the connected positions x(t) that satisfy the inequality in (A.9). This defines an interval of valid positions, $[x_1, x_2]$. The end–points of this interval satisfy $E_P = E_{tot}$. One of this points occurs for $-1 \leq x_1 < 0$. This is clear due to the form of $E_P(x)$, which behaves like $\frac{1}{2}x^2 + \frac{1}{3}|x|^3$ well for $x < 0$, so there is always a potential barrier at $x < 0$, which generates a positive force acting on the unit mass whenever $x < 0$:

$$-\frac{dE_P}{dx} > 0\,;\; x < 0 \tag{A.10}$$

From this, and noting that

$$E_P(x = -1) = \frac{1}{2} + \frac{1}{3}\epsilon > \frac{1}{2} - \frac{1}{3}\epsilon \tag{A.11}$$

we note that the point $x = -1$ is only attainable when $\epsilon = 0$ – the unit mass bounces back from the potential wall before reaching this point.

The potential energy E_P has a maximum at $x_{max} = 1/\epsilon$. If this maximum occurs at a point at the right of the initial position of the unit mass, $x_{max} > 1$, then the unit mass will always bounce back from the potential well. Thus, as long as $1/\epsilon > 1$ we have that:

$$-1 \leq x(t) \leq 1 \tag{A.12}$$

Noting that the minimum value of x(t) is strictly higher than -1 when $0 < \epsilon < 1$, we obtain:

$$-1 < x(t) \leq 1\,;\ \text{for } 0 < \epsilon < 1 \tag{A.13}$$

These equations imply the boundedness of $V(\phi)$, by the replacements $t \rightarrow \phi$ and $x \rightarrow V$.

$$-1 \leq V(\phi) \leq 1 \tag{A.14}$$

and

$$-1 < V(\phi) \leq 1\,;\ \text{for } 0 < \epsilon < 1 \tag{A.15}$$

RAY TRACING USING $\Omega(\epsilon)$

Consider an observer A immersed in a background distribution of far away light sources. This observer can obtain the angular position of every object in the sky, and determine the intensity of light that comes from every point in the sky, $I_A(\theta, \phi)$, in spherical coordinates. Now, imagine another point in space, B, far enough from the observer A such that the intensity of light that comes from every point in the sky, according to an observer in point B, is also given by the distribution found by observer A: $I_B(\theta, \phi) = I_A(\theta, \phi)$. If we place a black hole at point B, then light coming from the faraway sources will bend around the black hole such that the original observer will see a different distribution of light around the black hole. In this condition, the observer will be able to note that the black hole effectively subtends a solid angle in the sky – region in the sky devoid of any light due to the black hole. One half of the angle subtended by the black hole will effectively give the "radius" of the black hole, as seen by the observer, r_{BH}.

If we consider that the black region of the sky due to the black hole is due to the radius of the photon sphere, r_c, instead of the the Schwarzschild radius, r_s [1]. If we choose the coordinate system such that the black hole is at the positive x-axis, $\theta = \pi/2$ and $\phi = 0$, then, by definition of r_{BH}, the new distribution of light measured by the observer will obey (for small enough r_{BH}):

$$I(\theta, \phi) = 0\,;\; (\theta - \pi/2)^2 + \phi^2 \leq r_{BH} \tag{B.1}$$

[1]One can convince himself of this by considering the light from faraway objects that grazes the black hole at a distance given by $r = b$. This trajectory of this light is bended by the black hole for $b > r_c$. However, if $b < r_c$, the light will not escape and effectively no light coming from faraway objects will seem to originate from $r < r_c$, which becomes an effective radius for the black hole, according to observer A in these conditions. In the case that mass enters the black hole, and emits light from an r that obeys $r_s < r < r_c$, the light *can* escape the black hole, and is severely red-shifted. However, we are here considering a black hole with no light sources between $r_s < r < r_c$.

For other values of (θ, ϕ), the observer sees light distribution shifted by the $\Omega(\epsilon)$, where ϵ is given by:

$$\epsilon = \frac{r_c}{b} = \frac{r_{BH}}{\sqrt{(\theta - \pi/2)^2 + \phi^2}} \tag{B.2}$$

for $\theta \approx \pi/2$. We can use a further simplification of this latter equation, and use the coordinates (θ_x, θ_y) defined by $\theta_x = \phi$, $\theta_y = \theta - \pi/2$. For small values of θ_x and θ_y, say, in the order of milliradians, we can write:

$$I(\theta_x, \theta_y) = 0\,;\; \theta_x^2 + \theta_y^2 \leq r_{BH} \tag{B.3}$$

and

$$\epsilon = \frac{r_{BH}}{\sqrt{\theta_x^2 + \theta_y^2}} \tag{B.4}$$

where the analogue with Cartesian coordinates is evident. This coordinate system is shown in Figure B.1 for a black hole that subtends $4\pi \times 10^{-6}$ steradians, such that $r_{BH} = 2$ mrad.

This coordinate choice allows one to define the distribution of intensities that observer A sees to be (disregarding some attenuation factors):

$$I(\theta_x, \theta_y) = I_A\left(\theta_x - \Omega(\epsilon)\frac{\theta_x}{\sqrt{\theta_x^2 + \theta_y^2}}, \theta_y - \Omega(\epsilon)\frac{\theta_y}{\sqrt{\theta_x^2 + \theta_y^2}}\right)\;;\; \theta_x^2 + \theta_y^2 > r_{BH} \tag{B.5}$$

where we have used $I_A(\theta_x, \theta_y)$, the angular distribution of intensities seen by observer A without the black hole present, and using the coordinates (θ_x, θ_y). We can see the effect of applying equation (B.5) by using the $I_A(\theta_x, \theta_y)$ defined from Figure B.2.

To model the deflection of light with the distribution in Figure B.2, we use Equation (B.5) with $\Omega(\epsilon)$ approximated as a truncated first-degree Taylor polynomial, and as diagonal Padé approximants with $N = 2$ and $N = 10$. The resulting images can be found in Figures B.3 to B.5. We use a black hole with $r_{BH} = 10$ mrads.

The most notable difference between Figures B.3 and B.4 is the position of the white ring around the black hole, corresponding to the gravitational lensing of the big, white star at the black hole position $\theta_y = 0$. When using a better approximation of $\Omega(\epsilon)$, this ring has greater inner and outer radii, and is thinner. In Figure B.5, there are 8 white

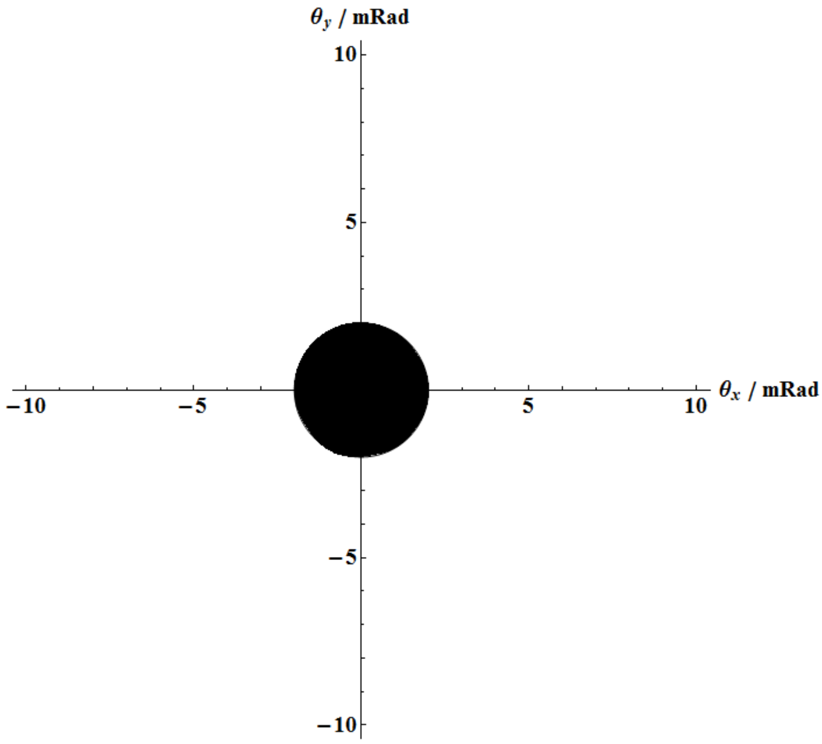

FIGURE B.1: A black hole with $r_{BH} = 2$ mrad in the center of the (θ_x, θ_y) coordinate system.

pixels around $\theta_x^2 + \theta_y^2 = 10$ millirads, corresponding to a second ring of light due to the big, white star.

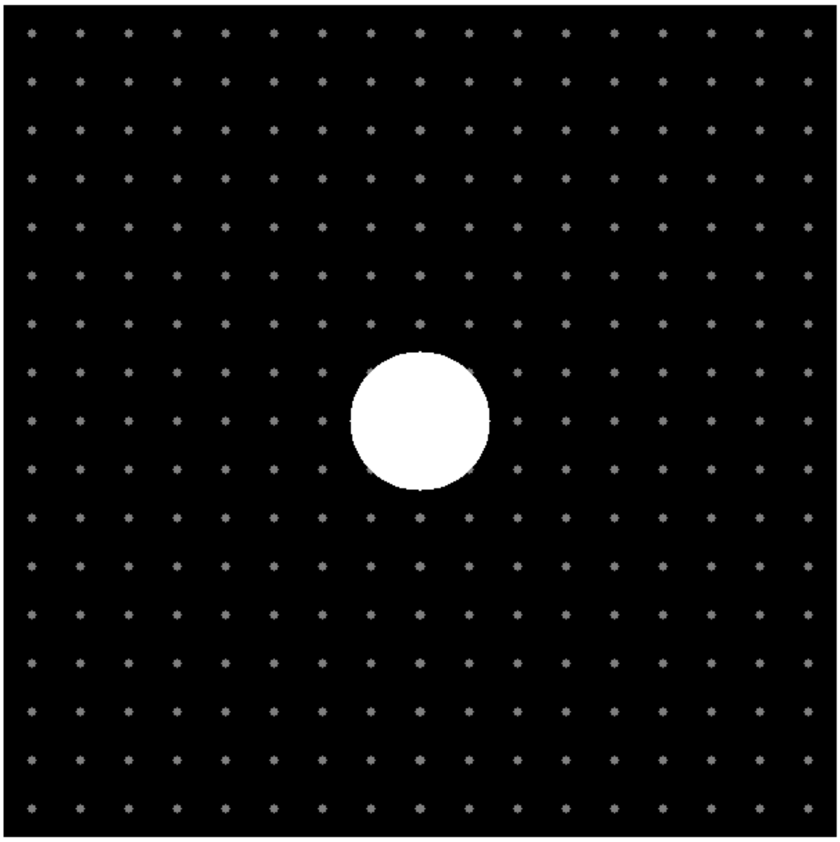

FIGURE B.2: 600×600 image corresponding to the intensity due to background light sources, $I_A(\theta_x, \theta_y)$, without a black hole present. Each pixel corresponds to 1 mrad. The big star, in white, has a radius of 50 mrad. The small stars, in gray, have a radius of 3 mrad. The star is at the center of the coordinate system, $(\theta_x, \theta_y) = (0, 0)$.

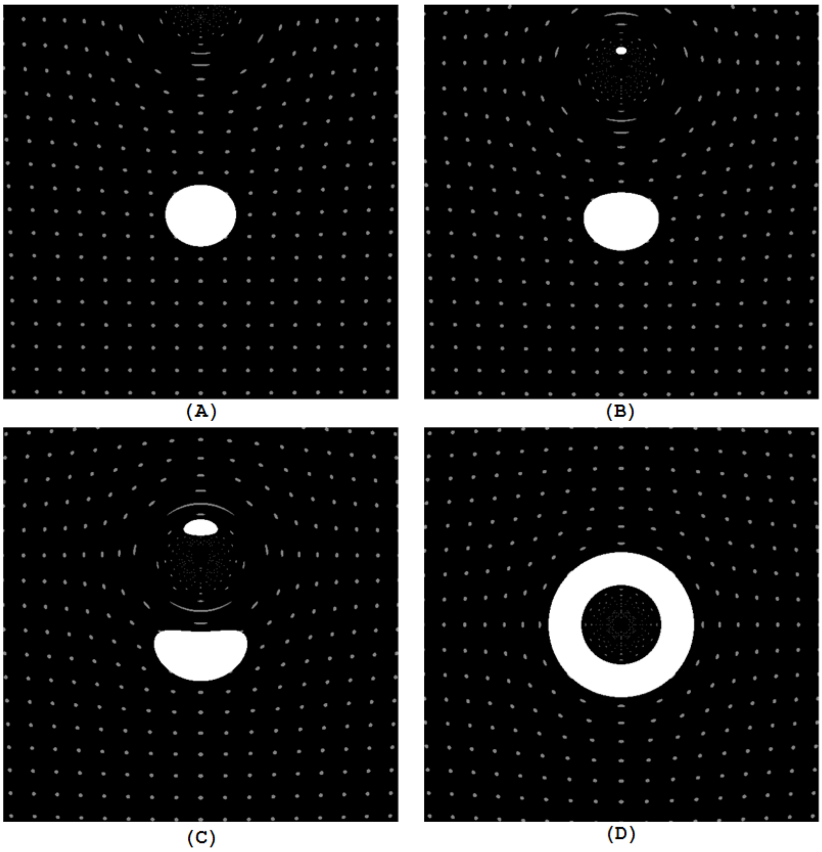

FIGURE B.3: Background of Figure B.2 warped by a black hole at (A) $\theta_y = 300$ mrad, (B) $\theta_y = 200$ mrad, (C) $\theta_y = 100$ mrad, and (D) $\theta_y = 0$ mrad. We make use of the truncated first-order Taylor polynomial of $\Omega(\epsilon)$.

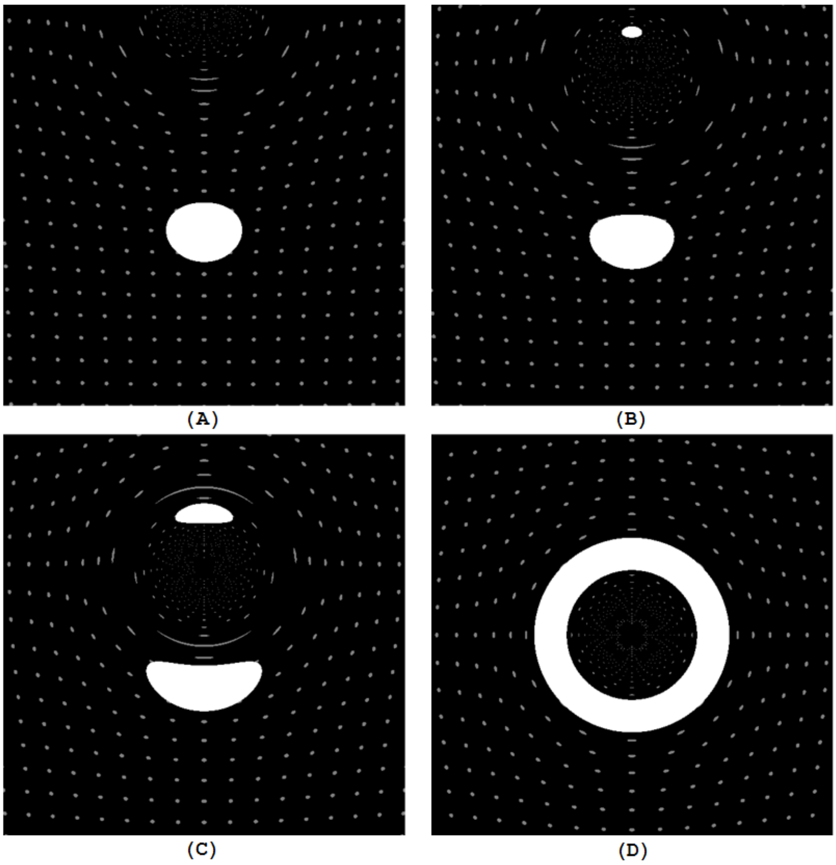

FIGURE B.4: Background of Figure B.2 warped by a black hole at (A) $\theta_y = 300$ mrad, (B) $\theta_y = 200$ mrad, (C) $\theta_y = 100$ mrad, and (D) $\theta_y = 0$ mrad. We make use of the diagonal $N = 2$ Padé approximant of $\Omega(\epsilon)$.

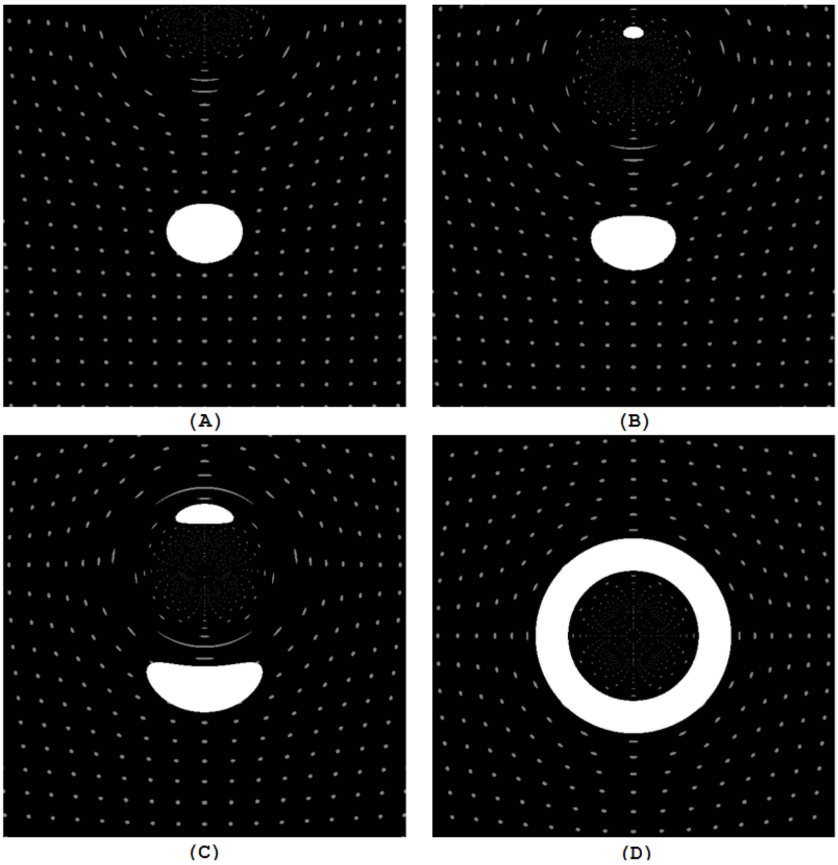

FIGURE B.5: Background of Figure B.2 warped by a black hole at (A) $\theta_y = 300$ mrad, (B) $\theta_y = 200$ mrad, (C) $\theta_y = 100$ mrad, and (D) $\theta_y = 0$ mrad. We make use of the diagonal $N = 10$ Padé approximant of $\Omega(\epsilon)$.

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